Anisotropic Best $\tau_C$-Approximation in Normed Spaces

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Abstract

The notions of the $\tau_C$-Kolmogorov condition, the $\tau_C$-sun and the $\tau_C$-regular point are introduced, and the relationships between them and the best $\tau_C$-approximation are explored. As a consequence, characterizations of best $\tau_C$-approximations from some kind of subsets (not necessarily convex) are obtained. As an application, a characterization result for a set $C$ to be smooth is given in terms of the $\tau_C$-approximation.

Keywords: Minkowski function; the best $\tau_C$-approximation; the $\tau_C$-Kolmogorov condition; the $\tau_C$-sun; smoothness.

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1 Introduction

Let $X$ be a real normed linear space and $C$ be a closed, bounded, convex subset of $X$ having the origin as an interior point. Recall that the Minkowski function $p_C$ with respect to the set $C$ is defined by

$$p_C(x) = \inf\{t > 0 : x \in tC\}, \ \forall \ x \in X.$$  \hspace{1cm} (1.1)

Let $G$ be a subset of $X$ and $x \in X$. Following [2], define the minimal time function $\tau_C(\cdot; G)$ by

$$\tau_C(x; G) := \inf_{g \in G} p_C(g - x), \ \forall x \in X.$$  \hspace{1cm} (1.2)

The study of the minimal time function $\tau_C(\cdot; G)$ is motivated by its worldwide applications in many areas of variational analysis, optimization, control theory, approximation theory, etc., and has received a lot of attention; see e.g., [2,6,7,9,11–13,17,18]. In particular, the proximal subgradient of the minimal time function $\tau_C(\cdot; G)$ is estimated and computed in [7, 17, 18], for Hilbert spaces with applications to control theory; while other various subgradients such as the Fréchet subgradients, Clark subgradients, the $\varepsilon$-subdifferential as well as the limiting subdifferential of the minimal time function $\tau_C(\cdot; G)$ in Hilbert spaces and/or general Banach spaces are explored in [6,9,13].

Our interest in the present paper is focused on the following minimization problem, denoted by $\min(x, G)$,

$$\min_{g \in G} p_C(g - x),$$  \hspace{1cm} (1.3)

where $x \in X$. Clearly, $g_0 \in G$ is a solution of the problem $\min(x, G)$ if and only if

$$p_C(g_0 - x) = \tau_C(x; G).$$

According to [2], any solution of the problem $\min(x, G)$ is called a best $\tau_C$-approximation (or, generalized best approximation) to $x$ from $G$. We denote by $P_C(x)$ the set of all best $\tau_C$-approximations to $x$ from $G$. The generic well-posedness of the minimization problem $\min(x, G)$ in terms of the Baire category was studied in [2,11], while the relationships between the existence of solutions and directional derivatives of the function $\tau_C(x; G)$ was explored in [12].

In the special case when $C$ is the closed unit ball $B$ of $X$, the minimal time function (1.2) and the corresponding minimization problem (1.3) are reduced to the distance function of $C$ and to the classical best approximation, respectively, which has been studied extensively and deeply, see, e.g., [3,16,21].

One aim of the present paper is to characterize the class of subsets of $X$ for which the so-called $\tau_C$-Kolmogorov condition holds about best $\tau_C$-approximations. Another aim of the present paper is to prove the equivalence between the smoothness of the underlying set $C$ and the convexity of $\tau_C$-$B$-suns. In particular, by taking $C$ to be the closed unit ball of $X$, our results extend the corresponding ones for nonlinear approximation problems; see, e.g., [3,5,21].
2 Preliminaries

Let $X$ be a real normed linear space and let $X^*$ denote its topological dual. Let $A$ be a nonempty subset of $X$. As usual, we use $\text{bd} A$ and $\text{int} A$ to denote respectively the boundary and the interior of $A$. The polar of $A$ is denoted by $A^\circ$ and defined by

$$A^\circ = \{ x^* \in X^* : x^*(x) \leq 1, \ \forall x \in A \}.$$  

Then $A^\circ$ is a weakly*-closed convex subset of $X^*$. Furthermore, in the case when $A$ is a convex bounded set with $0 \in \text{int} A$, $A^\circ$ is weakly*-compact with $0 \in \text{int} A^\circ$. In particular, $B^\circ$ equals the closed unit ball of $X^*$. Moreover, for a set $A \subseteq X^*$, $\text{ext} A$ and $A^\ast$ stand for the set of all extreme points and the weak* closure of $A$, respectively. The following proposition is exactly the well-known Krein-Milman Theorem, see, e.g., [10].

**Proposition 2.1.** Suppose that $A$ is a compact convex subset of $X^*$. Then $A$ equals the closed convex closure of $\text{ext} A$.

Let $\mu = \inf_{\|x\|=1} p_C(x)$ and $\nu = \sup_{\|x\|=1} p_C(x)$. We end this section with some known and useful properties of the Minkowski function; see [15, Section 1] for assertions (i)-(v) while (vi) is an immediate consequence of (iii) and (v).

**Proposition 2.2.** Let $x, y \in X$ and $x^* \in X^*$. Then we have the following assertions.

(i) $p_C(x) \geq 0$, and $p_C(x) = 0$ if and only if $x = 0$.

(ii) $p_C(x + y) \leq p_C(x) + p_C(y)$.

(iii) $p_C(\lambda x) = \lambda p_C(x)$ for each $\lambda > 0$.

(iv) $p_C(x) \leq 1$ if and only if $x \in C$.

(v) $p_C(x) = \sup_{x^* \in C^\circ} x^*(x)$ and $p_C^\circ(x^*) = \sup_{x \in C} x^*(x)$.

(vi) $\mu \|x\| \leq p_C(x) \leq \nu \|x\|$.

3 Characterization of the best $\tau_C$-approximation

The notion of suns introduced by Efimov and Stechkin (cf. [8]) has proved to be rather important in nonlinear approximation theory; see, e.g., [3–5,8,21] and references therein. In the following definition we extend this notion to the case of the generalized approximation. Throughout the whole paper, we always assume that $G \subseteq X$ is a nonempty subset of $X$, and let $x \in X$ and $g_0 \in G$, unless specially stated.

**Definition 3.1.** The element $g_0$ is called

(a) a $\tau_C$-solar point of $G$ with respect to $x$ if $g_0 \in P_G^C(x)$ implies that $g_0 \in P_G^C(x_{\lambda})$ for each $\lambda > 0$, where $x_{\lambda} = g_0 + \lambda(x - g_0)$;

(b) a $\tau_C$-solar point of $G$ if $g_0$ is a $\tau_C$-solar point of $G$ with respect to each $x \in X$.

We say that $G$ is a $\tau_C$-sun of $X$ if each point of $G$ is a $\tau_C$-solar point of $G$.  

Remark 3.1. We always write
\[ x_\lambda = g_0 + \lambda(x - g_0), \quad \forall \lambda > 0 \]
if no confusion caused. Thus
\[ p_C(g_0 - x_\lambda) = \lambda p_C(g_0 - x), \quad \forall \lambda > 0. \]

Furthermore, the following implication holds (cf. [13, Lemma 3.1]):
\[ g_0 \in P_G^C(x) \implies g_0 \in P_G^C(x_\lambda), \quad \forall \lambda \in [0, 1]. \]

For two points \( x, y \in X \), we use \([x, y]\) to denote the closed interval with ends \( x \) and \( y \), that is,
\[ [x, y] := \{tx + (1-t)y : t \in [0, 1]\}. \]

Recall (cf. [1]) that \( g_0 \in G \) is a star-shaped point of \( G \) if \([g_0, g] \subseteq G \) for each \( g \in G \). If \( G \) has a star-shaped point \( g_0 \), then \( G \) is called a star-shaped set with vertex \( g_0 \). Clearly, a convex subset is a star-shaped set with each vertex \( g_0 \in G \). We use \( S(g_0, G) \) to denote the star-shaped set with vertex \( g_0 \) generated by \( G \), that is,
\[ S(g_0, G) := \bigcup_{g \in G} [g_0, g]. \]

Proposition 3.1. Suppose that \( g_0 \in G \) is a star-shaped point of \( G \). Then \( g_0 \) is a \( \tau_C \)-solar point of \( G \). Consequently, any convex subset of \( X \) is a \( \tau_C \)-sun of \( X \).

Proof. Let \( x \in X \) be such that \( g_0 \in P_G^C(x) \). By Remark 3.1, we only need to show that \( g_0 \in P_G^C(x_\lambda) \) for each \( \lambda > 1 \). To this end, let \( \lambda > 1 \) and \( g \in G \). Since \( g_0 \) is a star-shaped point of \( G \), it follows from Proposition 2.2(iii) that
\[ p_C(g_0 - x) \leq p_C \left( \left( \left( 1 - \frac{1}{\lambda} \right) g_0 + \frac{1}{\lambda} g \right) - x \right) = \frac{1}{\lambda} p_C(g - x_\lambda). \]

By (3.2),
\[ p_C(g_0 - x_\lambda) = \lambda p_C(g_0 - x) \leq p_C(g - x_\lambda). \]

Hence, \( g_0 \in P_G^C(x_\lambda) \). This shows that \( g_0 \) is a \( \tau_C \)-solar point of \( G \). \( \square \)

The following proposition gives an equivalent condition for \( \tau_C \)-solar points in terms of star-shaped points.

Proposition 3.2. The element \( g_0 \) is a \( \tau_C \)-solar point of \( G \) if and only if
\[ g_0 \in P_G^C(x) \iff g_0 \in P_{S(g_0, G)}^C(x), \quad \forall x \in X. \]

Proof. For \( x \in X \), it is clear that \( g_0 \in P_{S(g_0, G)}^C(x) \implies g_0 \in P_G^C(x). \) Thus, to complete the proof, it suffices to verify that \( g_0 \) is a \( \tau_C \)-solar point of \( G \) if and only if
\[ g_0 \in P_G^C(x) \implies g_0 \in P_{S(g_0, G)}^C(x) \]
(3.4)
holds for each \( x \in X \). To this end, let \( x \in X \). By Definition 3.1 and Remark 3.1, one has that \( g_0 \) is a \( \tau_C \)-solar point of \( G \) if and only if the following implication holds:

\[
g_0 \in P_G^C(x) \implies g_0 \in P_G^C(x, \frac{1}{1-x}), \quad \forall \lambda \in [0, 1).
\]

Note by Proposition 2.2(iii) that

\[
g_0 \in P_G^C(x, \frac{1}{1-x}), \quad \forall \lambda \in [0, 1)
\]

\[
\iff \quad p_C(g_0 - x) \leq p_C((\lambda g_0 + (1 - \lambda)g) - x), \quad \forall g \in G, \quad \forall \lambda \in [0, 1)
\]

\[
\iff \quad g_0 \in P_G^C(S(x_0, G))(x).
\]

Hence, (3.5) holds if and only if (3.4) holds. This completes the proof. \(\square\)

**Definition 3.2.** The element \( g_0 \) is called a local best \( \tau_C \)-approximation to \( x \) from \( G \) if there exists an open neighborhood \( U(g_0) \) of \( g_0 \) such that \( g_0 \in P_G^C(x \cup U(g_0))(x) \).

Clearly, if \( g_0 \in P_G^C(x) \), then \( g_0 \) is a local best \( \tau_C \)-approximation to \( x \) from \( G \). The following proposition shows that the converse remains true if \( g_0 \) is a \( \tau_C \)-solar point of \( G \).

**Proposition 3.3.** Suppose that \( g_0 \) is a \( \tau_C \)-solar point of \( G \). Then \( g_0 \in P_G^C(x) \) if and only if \( g_0 \) is a local best \( \tau_C \)-approximation to \( x \) from \( G \).

**Proof.** The necessity part is clear as noted earlier. Below we prove the sufficiency part. To this end, suppose that \( g_0 \) is a local best \( \tau_C \)-approximation to \( x \) from \( G \). Then there exists an open neighborhood \( U(g_0) \) of \( g_0 \) such that \( g_0 \in P_G^C(x \cup U(g_0))(x) \). We claim that there is \( \lambda > 0 \) such that \( g_0 \in P_G^C(x_0) \). Indeed, otherwise, one has that \( g_0 \notin P_G^C(x_1/n) \) for each \( n \in \mathbb{N} \). Thus there exists a sequence \( \{g_n\} \subseteq G \) such that, for each \( n \in \mathbb{N} \),

\[
p_C(g_n - x_1/n) < p_C(g_0 - x_1/n) = \frac{1}{n}p_C(g_0 - x).
\]

(3.6)

It follows from Proposition 2.2(vi) that \( \lim_{n \to \infty} g_n = g_0 \). This implies that there exists \( n_0 \in \mathbb{N} \) such that \( g_{n_0} \in U(g_0) \cap G \). This together with (3.6) implies that \( g_0 \notin P_G^C(x_1/n_0) \), which is a contradiction by Remark 3.1 as \( g_0 \in P_G^C(x_0) \). Hence the claim stands; that is \( g_0 \in P_G^C(x_0) \) for some \( \lambda > 0 \). Noting that \( x = g_0 + \frac{1}{\lambda}(x_0 - g_0) \) and that \( g_0 \) is a \( \tau_C \)-solar point of \( G \), we have that

\[
g_0 \in P_G^C(g_0 + \frac{1}{\lambda}(x_0 - g_0)) = P_G^C(x)
\]

and completes the proof. \(\square\)

For the sequel study, we need to introduce the following notation:

\[
\Sigma_{g_0 - x} := \{x^* \in C^\circ : x^*(g_0 - x) = p_C(g_0 - x)\}.
\]

Then \( \Sigma_{g_0 - x} \) is a nonempty, weakly*-compact convex subset of \( C^\circ \). Furthermore, write \( \mathcal{E}_{g_0 - x} := \text{ext} \Sigma_{g_0 - x} \). Then \( \mathcal{E}_{g_0 - x} \neq \emptyset \) by Proposition 2.1. Moreover, by definition, we have that

\[
\mathcal{E}_{g_0 - x} = \text{ext} C^\circ \cap \Sigma_{g_0 - x}.
\]

(3.7)

The notions stated in Definition 3.3 below are extension of the notions of Kolmogorov Condition and Papini Condition in approximation theory to the setting of the best \( \tau_C \)-approximation theory, see, e.g., [4,5,21] and [14].
**Definition 3.3.** The pair \((x, g_0)\) is said to satisfy

(a) the \(\tau_C\)-Kolmogorov condition (the \(\tau_C\)-KC, for short) if

\[
\max_{x^* \in \Sigma_{g_0-x}} x^*(g - g_0) \geq 0, \quad \forall \ g \in G;
\]

(b) the \(\tau_C\)-Papini condition (the \(\tau_C\)-PC, for short) if

\[
\max_{x^* \in \Sigma_{g-x}} x^*(g_0 - g) \leq 0, \quad \forall \ g \in G.
\]

Clearly, by (3.7), \(\Sigma_{g_0-x}\) in (3.8) and \(\Sigma_{g-x}\) in (3.9) can be replaced by \(\mathcal{E}_{g_0-x}\) and \(\mathcal{E}_{g-x}\), respectively.

In the case when \(C\) is the closed unit ball of \(X\), that is, \(p_C\) is the norm, the notions of the regular point and the strongly regular point were introduced and studied respectively in [5] and [21]. The following notions of the \(\tau_C\)-regular point and the strongly \(\tau_C\)-regular point are respectively generalizations of the corresponding regular point and strongly regular point in best approximation theory.

**Definition 3.4.** The element \(g_0\) is called

(a) a \(\tau_C\)-regular point of \(G\) with respect to \(x\) if for any weakly\(^*\)-closed subset \(A\) of \(C^0\) satisfying for some \(g \in G\) the condition

\[
\mathcal{E}_{g_0-x} \subseteq A \subseteq \overline{\text{ext}C^0}\quad \text{and} \quad \min_{x^* \in \mathcal{E}} x^*(g_0 - g) > 0,
\]

there exists a sequence \(\{g_n\}\) such that \(g_n \to g_0\) and

\[
x^*(g_0 - g_n) > x^*(g_0 - x) - p_C(g_0 - x), \quad \forall \ x^* \in A, \ \forall \ n \in \mathbb{N};
\]

(b) a strongly \(\tau_C\)-regular point of \(G\) with respect to \(x\) if for any weakly\(^*\)-closed subset \(A\) of \(C^0\) satisfying (3.10) for some \(g \in G\), there exists a sequence \(\{g_n\}\) such that \(g_n \to g_0\) and

\[
x^*(g_0 - g_n) > 0, \quad \forall \ x^* \in A, \ \forall \ n \in \mathbb{N}.
\]

We say that \(G\) is a \(\tau_C\)-regular set (resp. strongly \(\tau_C\)-regular set) of \(X\) if each point of \(G\) is a \(\tau_C\)-regular point (resp. strongly \(\tau_C\)-regular point) of \(G\) with respect to each \(x \in X\).

Roughly speaking, a \(\tau_C\)-regular point \(g_0\) of \(G\) with respect to \(x\) means that, for any weakly\(^*\)-closed neighbourhood \(A\) of the extreme set \(\mathcal{E}_{g_0-x}\), if there exists some \(g \in G\) such that \(g_0 - g\) is positive on \(A\), then any neighbourhood of \(g_0\) contains an element \(g' \in G\) such that \(g_0 - g'\) has the same sign as \(g_0 - g\) on the set \(\mathcal{E}_{g_0-x}\) while out of this set it can have opposite sign but have to be controlled within \(p_C(g_0 - x) - x^*(g_0 - x)\); while, for a strongly \(\tau_C\)-regular point \(g_0\), \(g_0 - g'\) must have the same sign as \(g_0 - g\) on the whole weakly\(^*\)-closed neighbourhood \(A\).

**Remark 3.2.** Clearly, the strong \(\tau_C\)-regularity implies the \(\tau_C\)-regularity. We don’t know whether the converse is true even in the case when \(C\) is the closed unit ball of \(X\).

**Remark 3.3.** Clearly, any interior point of \(G\) is a strongly \(\tau_C\)-regular point of \(G\). Moreover, it is not difficult to check that any star-shaped point of \(G\) is a strongly \(\tau_C\)-regular point of \(G\).
Below we provide an example to illustrate the notions.

**Example 3.1.** Let \( X := \mathbb{R}^2 \) be the 2-dimensional Euclidean space. Let \( C \) be the equilateral triangle with the vertexes \((0, 2), (\sqrt{3}, -1)\) and \((-\sqrt{3}, -1)\) (see Figure 1). Then \( 0 \in \text{int} C \) and \( C^c \) is the equilateral triangle with vertexes \(x_1^*, x_2^*, x_3^*, \) where \(x_1^* := \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \ x_2^* := (0, -1)\) and \(x_3^* := \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right),\) and so \( \text{ext} C^c = \{x_1^*, x_2^*, x_3^*\} \) (see Figure 2). Let \( G \) be the subset defined by

\[
G = \left\{(t_1, t_2) : t_2 \geq \frac{1}{2} t_1, \ t_1 \leq 0\right\} \bigcup \left\{(t_1, t_2) : t_2 \geq -\frac{1}{2} t_1, \ t_1 \geq 0\right\}
\]

(see Figure 3). Clearly, \( G \) is not convex. We shall show that \( G \) is a strongly \( r_C \)-regular set. To this end, let \( g_0 \in G \). By Remark 3.3, we assume, without loss of generality, that \( g_0 = (a_1, a_2) \in \text{bd} G \) satisfies that

\[
a_1 > 0 \quad \text{and} \quad a_2 = -\frac{1}{2} a_1 \tag{3.13}
\]

(noting that the origin is a star-shaped point of \( G \)). Let \( A \) be a weakly*-closed subset of \( C^c \) satisfying (3.10) for some \( g = (b_1, b_2) \in G \) and some \( x \in X \). In the case when \( b_1 \geq 0 \) and \( b_2 \geq -\frac{1}{2} b_1 \), choose \( g_n = \left(1 - \frac{1}{n}\right) g_0 + \frac{1}{n} g \in G \) for each \( n \in \mathbb{N} \). Then it is easy to see that \( \{g_n\} \) is as desired. Below we consider the case when \( b_1 \leq 0 \) and \( b_2 \geq \frac{1}{2} b_1 \). It follows from (3.13) that

\[
(a_2 - b_2) - \frac{1}{2} (a_1 - b_1) = -a_1 + \frac{1}{2} b_1 - b_2 \leq -a_1 < 0.
\]

Consequently,

\[
x_n^*(g_0 - g) = -\frac{\sqrt{3}}{2} (a_1 - b_1) + \frac{1}{2} (a_2 - b_2) \leq -\frac{\sqrt{3}}{2} (a_1 - b_1) + \frac{1}{4} (a_1 - b_1) = \frac{3}{8} (1 - 2\sqrt{3}) a_1 < 0.
\]

This means that \( x_n^* \notin A \). Therefore, it suffices to consider the case when \( A = \{x_1^*, x_2^*\} \). To this end, let \( g_n = \left(1 - \frac{1}{n}\right) g_0 \) for each \( n \in \mathbb{N} \). Then \( \{g_n\} \subseteq G \) and \( g_n \to g_0 \). Noting that \( x_1^*(g_0) = \left(\frac{\sqrt{3}}{2} - \frac{1}{4}\right) a_1 \) and \( x_2^*(g_0) = \frac{1}{2} a_1 \), we have that

\[
\min_{x^* \in A} x^*(g_0 - g_n) = \frac{1}{n} \min_{x^* \in A} x^*(g_0) = \frac{1}{2n} a_1 > 0, \quad \forall n \in \mathbb{N}.
\]

This means that \( g_0 \) is a strongly regular point of \( G \), and so \( G \) is a strongly \( r_C \)-regular set.
Theorem 3.1. Consider the following assertion:

(i) The pair \((x, g_0)\) satisfies the \(\tau_C\)-KC.

(ii) The point \(g_0 \in \mathcal{P}_G^C(x)\).

(iii) The pair \((x, g_0)\) satisfies the \(\tau_C\)-PC.

Then (i)\(\Rightarrow\)(ii)\(\Rightarrow\)(iii). In addition, if \(g_0\) is a strongly \(\tau_C\)-regular point of \(G\) with respect to \(x\), then (i)\(\iff\)(ii)\(\iff\)(iii).

Proof. (i)\(\Rightarrow\)(ii). Suppose that (i) holds and let \(g \in G\). Then by Definition 3.3 (a), there exists \(x^* \in \Sigma_{g_0-x}\) such that \(x^*(g - g_0) \geq 0\). This together with Proposition 2.2 (v) implies that

\[
p_C(g_0 - x) = x^*(g_0 - g) + x^*(g - x) \leq x^*(g - x) \leq p_C(g - x);
\]

hence, \(g_0 \in \mathcal{P}_G^C(x)\) as \(g \in G\) is arbitrary and (ii) is proved.

(ii)\(\Rightarrow\)(iii). Suppose that (ii) holds. Let \(g \in G\) and \(x^* \in \Sigma_{g-x}\). Then \(x^*(g - x) = p_C(g - x)\); hence

\[
x^*(g_0 - g) = x^*(g_0 - x) + x^*(x - g) \leq p_C(g_0 - x) - p_C(g - x) \leq 0.
\]

This shows that \((x, g_0)\) satisfies the \(\tau_C\)-PC and (iii) is proved.

Suppose that \(g_0\) is a strongly \(\tau_C\)-regular point of \(G\) with respect to \(x\). It suffices to prove the implication (iii)\(\Rightarrow\)(i). To this end, suppose on the contrary that \((x, g_0)\) does not satisfy the \(\tau_C\)-KC. Then there exist \(g \in G\) and \(\delta > 0\) such that

\[
\max_{x^* \in \Sigma_{g_0-x}} x^*(g - g_0) = -\delta.
\]

Let

\[
U = \{x^* \in \overline{\text{ext} C^*} : x^*(g - g_0) < -\frac{\delta}{2}\} \quad \text{and} \quad A = \overline{U^*}.
\]

Then \(A\) is a weakly*-closed subset of \(C^*\) satisfying \(E_{g_0-x} \subseteq A \subseteq \overline{\text{ext} C^*}\) and

\[
\min_{x^* \in A} x^*(g_0 - g) \geq \frac{\delta}{2} > 0.
\]

Since \(g_0\) is a strongly \(\tau_C\)-regular point of \(G\) with respect to \(x\), there exists a sequence \(\{g_n\} \subseteq G\) such that

\[
\lim_{n \to \infty} g_n = g_0
\]

and

\[
\min_{x^* \in A} x^*(g_0 - g_n) > 0, \quad \forall \ n \in \mathbb{N}.
\]

Let \(K = \overline{\text{ext} C^*} \setminus U\). Then \(K\) is the weakly*-compact subset of \(\overline{\text{ext} C^*}\). Moreover, we have that \(K \cap \Sigma_{g_0-x} = \emptyset\) because \(E_{g_0-x} \subset U\) by (3.14) and (3.15). Consequently,

\[
\alpha := \max_{x^* \in K} x^*(g_0 - x) < p_C(g_0 - x).
\]

Set

\[
\alpha_0 := \frac{1}{2}(p_C(g_0 - x) - \alpha).
\]
Thus, by the separation theorem (cf. [10]), there exist $\in (g, P)$ for the pair $(p, g)$. In particular, we have that $p \in (g, g)$. Without loss of generality, we may assume that $g = 0$. Since $\lim_{n \to \infty} g_n = g_0$ by (3.17), one has from Proposition 2.2 (vi) that there exists $n_0 \in \mathbb{N}$ such that

$$\max\{pC(g_{n_0} - g_0), pC(g_0 - g_{n_0})\} < \alpha_0.$$  

(3.21)

Thus

$$pC(g_0 - x) \leq pC(g_0 - g_{n_0}) + pC(g_{n_0} - x) < \alpha_0 + pC(g_{n_0} - x).$$

(3.22)

It follows from Proposition 2.2 (v), (3.19)-(3.22) that

$$\max_{x^* \in K} x^*(g_{n_0} - x) \leq \max_{x^* \in K} x^*(g_0 - g_0) + \max_{x^* \in K} x^*(g_0 - x) \leq pC(g_{n_0} - g_0) + \alpha \leq \alpha_0 + \alpha = pC(g_0 - x) - \alpha_0 < pC(g_{n_0} - x).$$

This shows that $K \cap E_{g_{n_0} - x} = \emptyset$, and so $E_{g_{n_0} - x} \subseteq \overline{C^0} \setminus K = U$. Therefore,

$$\max_{x^* \in E_{g_{n_0} - x}} x^*(g_0 - g_{n_0}) \geq \inf_{x^* \in U} x^*(g_0 - g_{n_0}) = \min_{x^* \in A} x^*(g_0 - g_{n_0}) > 0$$

thanks to (3.18). Thus, $\max_{x^* \in E_{g_{n_0} - x}} x^*(g_0 - g_{n_0}) > 0$ by (3.7). This contradicts the $\tau_C$-PC for the pair $(x, g_0)$, and the proof is complete. \hfill \Box

Remark 3.4. One natural and interesting question is: Wether the implication (iii)$\Rightarrow$(i) remains true under simple (not strong) regularity assumption? Even in the case when $C$ is the unit closed ball, we don’t know the answer and so we leave it open.

The main result of this section is a generalization of [4, Theorem 9].

Theorem 3.2. The following assertions are equivalent.

(i) The element $g_0$ is a $\tau_C$-solar point of $G$ with respect to $x$.

(ii) The element $g_0 \in P_C^C(x)$ if and only if $(x, g_0)$ satisfies the $\tau_C$-$KC$.

(iii) The element $g_0$ is a $\tau_C$-regular point of $G$ with respect to $x$.

Proof. (i)$\Rightarrow$(ii). Suppose that (i) holds. The sufficiency part of (ii) follows directly from Theorem 3.1. Below we show the necessity part of (ii). To this end, we assume that $g_0 \in P_C^C(x)$. Without loss of generality, we may assume that $x \neq g_0$ and $pC(g_0 - x) \neq 0$. Let $g \in G \setminus \{g_0\}$ be arbitrary. Then $g_0 \in P_C^C([g_0, g], [g_0, g]) \cap \text{int}(C + x) = \emptyset$. Thus, by the separation theorem (cf. [10]), there exist $y^* \in X^* \setminus \{0\}$ and a real number $r$ such that

$$y^*(z - x) \geq r, \quad \forall z \in [g_0, g]$$

(3.23)

and

$$y^*(y - x) \leq r, \quad \forall y \in C + x.$$ 

(3.24)

In particular, we have that $pC(y^*) \leq r$ by Proposition 2.2(v). Let $x^* = r^{-1}y^*$. Then $pC(x^*) \leq 1$ and so $x^* \in C^0$. Since $g_0 \in [g_0, g] \cap (C + x)$, it follows from (3.23) and (3.24) that

$$r = y^*(g_0 - x).$$

(3.25)
This implies that \( x^*(g_0 - x) = 1 = p_C(g_0 - x) \), and so \( x^* \in \Sigma_{g_0 - x} \). Furthermore, \( x^*(g - g_0) \geq 0 \) thanks to (3.23) and (3.25). Hence the necessity part holds as \( g \in G \setminus \{g_0\} \) is arbitrary and the implication is proved.

(ii) \( \Rightarrow \) (i). Suppose that (ii) holds and assume that \( g_0 \in P_G^C(x) \). Then
\[
\max_{x^* \in \Sigma_{g_0 - x}} x^*(g - g_0) \geq 0, \quad \forall \ g \in G. \tag{3.26}
\]
Let \( \lambda > 0 \) be arbitrary. Noting that \( g_0 - x_{\lambda} = \lambda(g_0 - x) \), one has that \( \Sigma_{g_0 - x_{\lambda}} = \Sigma_{g_0 - x} \). This together with (3.26) implies that \( (x_{\lambda}, g_0) \) satisfies the \( \tau_C \)-KC, and so \( g_0 \in P_G^C(x_{\lambda}) \). This shows that \( g_0 \) is a \( \tau_C \)-solar point of \( G \) with respect to \( x \).

(ii) \( \Rightarrow \) (iii). Suppose that (ii) holds. Let \( g \in G \) and \( A \) be a weakly\(^*\)-closed subset of \( C^o \) satisfying (3.10). Then
\[
\max_{x^* \in x_{\lambda} - x} x^*(g - g_0) \leq \max_{x^* \in A} x^*(g - g_0) < 0.
\]
This together with (3.7) implies that \( (x, g_0) \) does not satisfy the \( \tau_C \)-KC; hence, \( g_0 \notin P_G^C(x) \) by (ii). Since \( g_0 \) is a \( \tau_C \)-solar point of \( G \) with respect to \( x \) by the equivalence of (i) and (ii) just proved, it follows from Proposition 3.3 that \( g_0 \) is not local best \( \tau_C \)-approximation to \( x \) from \( G \). Thus there exists a sequence \( \{g_n\} \subseteq G \) such that \( g_n \to g_0 \) and
\[
p_C(g_n - x) < p_C(g_0 - x), \quad \forall \ n \in \mathbb{N}. \tag{3.27}
\]
Let \( x^* \in A \) be arbitrary. Then, \( x^*(g_n - x) \leq p_C(g_n - x) \) for each \( n \in \mathbb{N} \). This and (3.27) imply that
\[
x^*(g_0 - g_n) \geq x^*(g_0 - x) - p_C(g_n - x) > x^*(g_0 - x) - p_C(g_0 - x), \quad \forall \ n \in \mathbb{N}.
\]
In view of Definition 3.4, \( g_0 \) is a \( \tau_C \)-regular point of \( G \).

(iii) \( \Rightarrow \) (ii). Suppose that (iii) holds. By Theorem 3.1 (i), it suffices to prove that \( (x, g_0) \) satisfies the \( \tau_C \)-KC whenever \( g_0 \in P_G^C(x) \). To this end, we assume on the contrary that it is not the case. Then, there are \( g \in G \) and \( \delta > 0 \) such that (3.14) holds. Let \( U \) and \( A \) be defined by (3.15). Then (3.16) holds. Since \( g_0 \) is a \( \tau_C \)-regular point of \( G \) with respect to \( x \), there exists a sequence \( \{g_n\} \subseteq G \) such that \( \lim_{n \to \infty} g_n = g_0 \) and (3.11) holds. This together with Proposition 2.2 implies that
\[
\max_{x^* \in A} x^*(g_n - x) < p_C(g_0 - x), \quad \forall \ n \in \mathbb{N}. \tag{3.28}
\]
On the other hand, let \( K = \text{ext} C^o \setminus U \). Then there is \( \epsilon > 0 \) such that
\[
\max_{x^* \in K} x^*(g_0 - x) < p_C(g_0 - x) - \epsilon. \tag{3.29}
\]
Since \( \lim_{n \to \infty} g_n = g_0 \), one has that \( \lim_{n \to \infty} p_C(g_n - g_0) = 0 \). Let \( n_0 \in \mathbb{N} \) be such that \( p_C(g_{n_0} - g_0) < \epsilon \). It follows from (3.29) that
\[
\max_{x^* \in K} x^*(g_{n_0} - x) \leq p_C(g_{n_0} - g_0) + \max_{x^* \in K} p_C(g_0 - x) < p_C(g_0 - x). \tag{3.30}
\]
Combining (3.28) and (3.30), we obtain that \( p_C(g_{n_0} - x) < p_C(g_0 - x) \), which contradicts that \( g_0 \in P_G^C(x) \). The proof is complete.
The following corollary is a global version of Theorem 3.2.

**Corollary 3.1.** The following statements are equivalent.

(i) $G$ is a $\tau_C$-sun of $X$.

(ii) For each $g_0 \in G$ and each $x \in X$, $g_0 \in P_G^C(x)$ if and only if $(x, g_0)$ satisfies the $\tau_C$-$KC$.

(iii) $G$ is a $\tau_C$-regular set.

### 4 Smoothness and convexity of $\tau_C$-$B$-suns

We begin with the notion of smooth convex sets (cf. [10]). Let $x \in \text{bd}C$ and $x^* \in C^\circ$. Recall that $x^*$ is a supporting functional of $C$ at $x$ if $x^*(x) = 1$.

**Definition 4.1.** The set $C$ is called smooth if each point of $\text{bd}C$ has a unique supporting functional.

The following notion extends a similar concept introduced in [3].

**Definition 4.2.** The set $G$ is called a $\tau_C$-$B$-sun of $X$ if for each $x \in X$ there exists $g_0 \in P_G^C(x)$ such that $g_0$ is a $\tau_C$-solar point of $G$ with respect to $x$.

**Remark 4.1.** Clearly, if $G$ is an existence set (i.e., $P_G^C(x) \neq \emptyset$ for each $x \in X$), then a $\tau_C$-sun must be $\tau_C$-$B$-sun. The converse is not true in general, see [21, Example 1.4] for the case when $C$ is the unit ball of $X$.

The main result of this section is as follows, which extends [3, Theorem 2.5] to the setting of the best $\tau_C$-approximation.

**Theorem 4.1.** The set $C$ is smooth if and only if each $\tau_C$-$B$-sun of $X$ is convex.

**Proof.** “$\Longrightarrow$”. Suppose that $C$ is smooth and $G$ is a $\tau_C$-$B$-sun of $X$. It suffices to verify that $\frac{1}{2}(g_1 + g_2) \in G$ for each pair of elements $g_1, g_2 \in G$. To this end, let $g_1, g_2 \in G$ and let $x = \frac{1}{2}(g_1 + g_2)$. By Definition 4.2 there exists $g_0 \in P_G^C(x)$ such that $g_0$ is a $\tau_C$-solar point of $G$ with respect to $x$. It follows from Theorem 3.2 that there exist $x^*_1, x^*_2 \in \Sigma_{g_0 - x}$ such that

$$x^*_i(g_i - g_0) \geq 0, \quad \forall i = 1, 2. \tag{4.1}$$

Suppose that $g_0 \neq x$ and consider the point $\bar{y} := (g_0 - x)/p_C(g_0 - x)$. Then $\bar{y} \in C$ and $x^*_i(\bar{y}) = 1$ for each $i = 1, 2$. Noting that each $x^*_i \in C^\circ$, we have that $x^*_i, i = 1, 2$, are supporting functionals of $C^\circ$ at $\bar{y}$. Hence, $x^*_1 = x^*_2$ by the smoothness of $C$. Let $x^* = x^*_1$. Then

$$p_C(g_0 - x) = x^*(g_0 - x) = \frac{1}{2}x^*(g_0 - g_1) + \frac{1}{2}x^*(g_0 - g_2) \leq 0$$

thanks to (4.1). By Proposition 2.2 (i), one has $x = g_0$, which is a contradiction, and hence $\frac{1}{2}(g_1 + g_2) \in G$.

“$\Longleftarrow$”. Conversely, suppose on the contrary that $C$ is not smooth. Then there exist $x_0 \in \text{bd}C$ and two functional $x^*_1, x^*_2 \in C^\circ$ such that

$$x_1 \neq x_2 \quad \text{and} \quad x^*_1(x_0) = x^*_2(x_0) = 1. \tag{4.2}$$
Let $G_i := \{ x \in X : x_i^*(x) \geq 0 \}$ for each $i = 1, 2$, and let $G = G_1 \cup G_2$. Then $G$ is a closed and nonconvex subset of $X$. In fact, the closedness is clear. To prove the nonconvexity, we first prove that $\ker(x_1^*) \cap G_2 \neq \emptyset$, where $\ker(x^*) := \{ x \in X : x^*(x) = 0 \}$ is the kernel of the functional $x^*$. Indeed, otherwise, one has that $\ker(x_1^*) \subseteq G_2$. Let $x \in X$. Then, by (4.2),

$$x - x_i^*(x)x_0 \in \ker(x_i^*), \quad \forall \ i = 1, 2. \tag{4.3}$$

In particular, we have that $x - x_1^*(x)x_0 \in G_2$. It follows that

$$x_2^*(x) \geq x_2^*(x_0)x_1^*(x) = x_1^*(x), \quad \forall \ x \in X.$$ 

This implies that $x_1^* = x_2^*$, which is a contradiction. Therefore, $\ker(x_1^*) \cap G_2 \neq \emptyset$. Similarly, we also have that $\ker(x_2^*) \cap G_1 \neq \emptyset$. Take $x_1 \in \ker(x_1^*) \cap G_2$ and $x_2 \in \ker(x_2^*) \cap G_1$. Then $x_1, x_2 \in G$ and $x_i^*(\frac{1}{2}x_1 + \frac{1}{2}x_2) < 0$ for each $i = 1, 2$. This means that $\frac{1}{2}x_1 + \frac{1}{2}x_2 \notin G$, and so $G$ is not convex.

By the definition of $\tau_C$, it is easy to see that

$$\tau_C(x; G) = \min \{ \tau_C(x; G_1), \tau_C(x; G_2) \}, \quad \forall \ x \in X. \tag{4.4}$$

We will prove that, for each $i = 1, 2$,

$$\tau_C(x; G_i) = -x_i^*(x), \quad \forall \ x \in X \setminus G_i. \tag{4.5}$$

To this end, fix $i = 1, 2$ and let $x \in X \setminus G_i$. Then $x_i^*(x) < 0$, and by (4.3), one has that

$$\tau_C(x; G_i) \leq p_C((x - x_i^*(x)x_0) - x) = -x_i^*(x)p_C(x_0) = -x_i^*(x)$$

(noting that $p_C(x_0) = 1$). On the other hand,

$$\tau_C(x; G_i) = \inf_{g \in G_i} p_C(g - x) \geq \inf_{g \in G_i} x_i^*(g - x) \geq -x_i^*(x),$$

and the assertion (4.5) is seen to hold.

Below we prove that $G$ is a $\tau_C$-$B$-sum of $X$. To this end, let $x \in X \setminus G$ and $d := \tau_C(x; G)$. Without loss of generality, we may assume that

$$\tau_C(x; G_1) \leq \tau_C(x; G_2). \tag{4.6}$$

Then by (4.5),

$$d = \tau_C(x; G_1) = -x_1^*(x). \tag{4.7}$$

Let $g_0 = x + dx_0$. Then by (4.3)

$$g_0 = x - x_1^*(x)x_0 \in \ker(x_1^*) \subseteq G_1 \subseteq G. \tag{4.8}$$

Moreover

$$p_C(g_0 - x) = dp_C(x_0) = d = \tau_C(x; G_1) = \tau_C(x; G). \tag{4.9}$$

Hence $g_0 \in P^C_G(x)$. We assert that $g_0 \in P^C_G(x_\lambda)$ for each $\lambda > 0$. Granting this, $G$ is a nonconvex, $\tau_C$-$B$-sum and the proof of Theorem 4.1 is complete. Let $\lambda > 0$. By (4.8) and (4.9), we have that $g_0 \in P^C_G(x)$. Since $G_1$ is convex, it follows from Proposition 3.1 that
\( g_0 \in P_{G_1}^C(x_\lambda) \). Thus, to complete the proof, it suffices to show that \( \tau_C(x_\lambda; G) = \tau_C(x_\lambda; G_1) \).

To do this, note that \( x_2^*(x) \leq x_1^*(x) = -d \) by (4.5), (4.6) and (4.7). Consequently,

\[
\begin{align*}
x_2^*(x_\lambda) &= (1 - \lambda)x_2^*(g_0) + \lambda x_2^*(x) \\
&= (1 - \lambda)(x_2^*(x) + dx_2^*(x_0)) + \lambda x_2^*(x) \\
&= x_2^*(x) + d(1 - \lambda) \\
&\leq -d + d(1 - \lambda) \\
&= -\lambda d.
\end{align*}
\]

This together with (4.5) (with \( x_\lambda \) in place of \( x \)) implies that

\[
\tau_C(x_\lambda; G_2) \geq d\lambda = \tau_C(x_\lambda; G_1).
\]

Therefore, \( \tau_C(x_\lambda; G) = \tau_C(x_\lambda; G_1) \) thanks to (4.4). The proof is complete.

\[
\Box
\]

References


